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The coset D-branes of SU(n)

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Abstract: Using a nested coset construction a collection of D-branes that appear to generate all the K-theory charges of string theory on SU(n) are constructed and their charges are determined.

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1. Introduction

A lot of evidence has been accumulated over the last few years that the charges of D-branes can be described in terms of (twisted) K-theory [1, 2, 3]. For example, for the case of string theory on the simply connected Lie group G, the charge group is conjectured to be the twisted K-group of G (for more details see for example [4, 5]). Modulo some technical assumption, this twisted K-group has been calculated in [5] (see also [6, 7]) to be

$$ {}^{k+h^\vee}K^*(G) \cong \mathbb{Z}_d^{\oplus m}, \quad m = 2^{\mathrm{rk}(\bar{\mathfrak{g}})-1}, \tag{1.1} $$

where $h^\vee$ is the dual Coxeter number of G, k is the level of the underlying WZW model, and d is the integer

$$ d = \frac{k+h^\vee}{\gcd(k+h^\vee, L)}. \tag{1.2} $$

Here L only depends on $\bar{\mathfrak{g}}$, the finite dimensional Lie algebra associated to $\mathfrak{g}_k$. The summands $\mathbb{Z}_d$ are equally divided between even and odd degree if $\mathrm{rk}(\bar{\mathfrak{g}}) > 1$. For $\bar{\mathfrak{g}} = \mathfrak{su}(2)$ the only summand is in even degree. In this paper we shall only consider the case of $\mathrm{SU}(n)$, for which $\mathrm{rk}(\mathfrak{su}(n)) = n-1$, and thus the multiplicity is $m = 2^{n-2}$.

On the other hand, D-branes can be constructed in terms of the underlying conformal field theory, and it should be possible to determine their charges using this microscopic description. In particular, it was shown in [8] how the charges for branes that preserve the affine algebra $\mathfrak{g}_k$ (up to an automorphism) can in principle be calculated. For the D-branes that preserve the full algebra without any automorphism, the charges were then determined for $\mathrm{SU}(n)$ [8, 9], and later for all simply-connected Lie groups [10], and it was found that they account precisely for one summand $\mathbb{Z}_d$. The charges of the D-branes that preserve the affine algebra up to an outer automorphism were calculated in [11]; whenever these twisted D-branes exist, they also contribute one summand $\mathbb{Z}_d$. These D-branes therefore only account at most for two of the summands $\mathbb{Z}_d$ in (1.1).

Recently, a conformal field description of the remaining D-branes for SU(n) was proposed [12]. This construction was inspired by the suggestion of [9] that the remaining charges should be related by some sort of T-duality to the original untwisted and twisted branes. It was found that there are precisely 2^{n-2} different constructions that lead to boundary states that can be distinguished by their coupling to the bosonic (and fermionic) degrees of freedom associated to the Cartan torus — in particular, these D-branes therefore couple differently to the RR ground states of the theory. While these boundary states break in general the affine symmetry, their open string spectra could still be described in terms of twisted representations of the affine symmetry algebra. As a consequence their charges could be determined, and it was found that each of the 2^{n-2} different constructions leads to the same charge group $\mathbb{Z}_d$.

While this construction is quite suggestive, an obvious geometric interpretation of the different D-branes was not available. In this paper we construct a different collection of D-branes whose geometric interpretation is somewhat clearer. It is well known that the group SU(n) is rationally homotopy equivalent to a product of odd dimensional spheres

$$\mathrm{SU}(n) \cong S^{2n-1} \times S^{2n-3} \times \cdots \times S^3 \,. \tag{1.3}$$

Each of these spheres comes from a coset space

$$\mathrm{SU}(m)/\mathrm{SU}(m-1) \cong S^{2m-1} \,, \tag{1.4}$$

and thus, as regards rational homotopy equivalence, we can think of SU(n) as the product

$$\begin{aligned} \mathrm{SU}(n) \cong & \Big(\mathrm{SU}(n)/\mathrm{SU}(n-1)\Big) \times \Big(\mathrm{SU}(n-1)/\mathrm{SU}(n-2)\Big) \times \\ & \times \cdots \times \Big(\mathrm{SU}(3)/\mathrm{SU}(2)\Big) \times \mathrm{SU}(2) \,. \end{aligned} \tag{1.5}$$

Moreover, the homology of SU(n) is isomorphic to the homology of the sphere product $H_*(\mathrm{SU}(n);\mathbb{Z}) \cong H_*(S^{2n-1} \times \cdots \times S^3;\mathbb{Z})$. In terms of the conformal field theory description of strings on SU(n) this suggests that we should decompose the space of states with respect to the corresponding coset algebras (see also [13, 14]), i.e. with respect to the W-algebra[1]

$$\mathcal{W} = \widehat{\mathfrak{su}}(n)/\widehat{\mathfrak{su}}(n-1) \oplus \widehat{\mathfrak{su}}(n-1)/\widehat{\mathfrak{su}}(n-2) \oplus \cdots \oplus \widehat{\mathfrak{su}}(4)/\widehat{\mathfrak{su}}(3) \oplus \widehat{\mathfrak{su}}(3) \,. \tag{1.6}$$

The D-branes we construct respect this algebra, and they are therefore characterised by the gluing conditions that we impose for the different factors. In fact, there are two different gluing conditions that can be chosen for each factor, and this leads to 2^{n-2} different constructions. As we shall see, the construction of these boundary states (and in particular their open string spectrum) decouples for the various factors in $\mathcal{W}$. It is then suggestive to believe that the two constructions for each factor correspond essentially to the choice of whether or not the the homology class of the respective sphere contributes to the homology

[1] Because of the non-trivial B-flux, D-branes cannot wrap the sphere S^3; as regards the construction of D-branes we therefore do not need to break the symmetry of the last SU(3) factor.

class of the D-brane world volume.[2] Given the relation between K-theory and homology [9] (namely that the summands of (1.1) are in one to one correspondence with the 2^{n-2} different homology cycles that can be wrapped by D-branes), this then suggests that the above construction accounts for all K-theory charges.

Intriguingly, this construction is technically in many respects very similar indeed to the construction of [12]. This gives support to the assertion that the branes described here represent different points on the same sheets of moduli space as the branes of [12].

The paper is organised as follows. In the following section we describe the construction of the D-branes, generalising the construction of [13, 14]. We show that their open string spectrum gives rise to a NIM-rep of the product of coset algebras. Furthermore, we identify flows that reduce the charges that are carried by these D-branes to summands $\mathbb{Z}_d$. In section 3 we show, following the analysis of [14], that at least some of these branes wrap homologically inequivalent cycles. Section 4 contains some conclusions.

2. The inductive coset argument

String theory on a group manifold G is described in terms of representations of the affine Lie algebra $\mathfrak{g}$ at level k. For the situation where the group manifold is simply-connected (a systematic analysis of the D-brane charges on non simply-connected Lie groups was recently started in [15, 16, 17]), the full spectrum of the theory is then

$$\mathcal{H} = \bigoplus_{\lambda\in P^+(\mathfrak{g})} \mathcal{H}_\lambda \otimes \bar{\mathcal{H}}_{\lambda^*} , \tag{2.1}$$

where the sum runs over all integrable highest weight representations λ of $\mathfrak{g}_k$, and the representation for the right-movers is conjugate to the representation of the left-movers. This theory is therefore sometimes referred to as the 'charge-conjugation' theory. In this paper we shall deal with the case when $\bar{\mathfrak{g}} = \mathfrak{su}(n)$.

For $\bar{\mathfrak{g}} = \mathfrak{su}(2)$, the usual untwisted D-branes, i.e. the D-branes that preserve the affine symmetry without any automorphism,[3] generate already the full charge group $\mathbb{Z}_d$. The corresponding boundary states are labelled by the integrable highest weights $\rho \in P_2^+ := P^+(\mathfrak{su}(2)_k)$ (here as in the following the index '2' refers to the fact that we are dealing with $\mathfrak{su}(2)$, and we are suppressing the dependence on k), and they are explicitly given by the familiar Cardy formula

$$\|\rho_2\rangle\!\rangle = \sum_{\lambda_2\in P_2^+} \frac{S_{\rho_2\lambda_2}}{\sqrt{S_{0\lambda_2}}} |\lambda_2\rangle\!\rangle^{\mathrm{id}} , \tag{2.2}$$

[2]While we are not able to make this precise at present, we can show, following the analysis of [14], that at least some of these constructions correspond to branes wrapping homologically inequivalent cycles; this will be discussed in section 3.

[3]D-branes that preserve the affine symmetry up to an inner automorphism can be related to one another by the group action. All of these D-branes therefore carry the same charge; in the following we can thus ignore such inner automorphisms.

where $S_{\rho\lambda}$ is the modular S-matrix, and $|\lambda\rangle\rangle^{\rm id}$ denotes the canonically normalised Ishibashi state with trivial gluing conditions in the $\mathcal{H}_\lambda \otimes \bar{\mathcal{H}}_{\lambda^*}$ sector. The overlap between two such branes leads to open string multiplicities that are simply described by the fusion rules,

$$\langle\langle \rho_2 \| q^{L_0+\bar{L}_0-\frac{c}{12}} \| \rho_2' \rangle\rangle = \sum_{\lambda_2 \in P_2^+} N_{\rho_2\lambda_2}{}^{\rho_2'}\, \tilde{\chi}_{\lambda_2}\,, \tag{2.3}$$

where $\tilde{\chi}_\lambda$ is the character of the λ representation in the open string description.

For $\bar{\mathfrak{g}} = \mathfrak{su}(3)$, the analogue of the above construction (that we shall call the 'straight' construction in the following) exists, but there is also a second 'twisted' construction, which accounts for the second $\mathbb{Z}_d$-summand in the charge group $\mathbb{Z}_d \oplus \mathbb{Z}_d$. Namely, $\mathfrak{su}(3)$ has a non-trivial outer automorphism ω corresponding to complex conjugation, which can be used to twist the gluing conditions. Boundary conditions obeying ω-twisted gluing conditions are labelled by the twisted representations x_3 of $\widehat{\mathfrak{su}}(3)$, and the corresponding boundary states are explicitly given as

$$\| x_3 \rangle\rangle = \sum_{\lambda_3 \in P_3^-} \frac{\psi_{x_3\lambda_3}}{\sqrt{S_{0\lambda_3}}}\, |\lambda_3\rangle\rangle^\omega\,, \tag{2.4}$$

where the ψ-matrix can be identified with a twisted S-matrix (see for example [18]), and $\lambda_3 \in P_3^-$ denotes the charge conjugation invariant weights of $\widehat{\mathfrak{su}}(3)_k$, which give rise to ω-twisted Ishibashi states $|\lambda_3\rangle\rangle^\omega$. These D-branes lead to open string multiplicities of the form

$$\langle\langle x_3 \| q^{L_0+\bar{L}_0-\frac{c}{12}} \| x_3' \rangle\rangle = \sum_{\lambda_3 \in P_3^+} \mathcal{N}_{x_3\lambda_3}{}^{x_3'}\, \tilde{\chi}_{\lambda_3}\,, \tag{2.5}$$

where $\mathcal{N}_{x\lambda}{}^{x'}$ are the twisted fusion rules.

For SU(2) and SU(3) these constructions already produce D-branes that generate the full K-theory groups. However, for SU(n) with $n \geq 4$, the K-theory analysis suggests a charge group consisting of 2^{n-2} $\mathbb{Z}_d$-subgroups, whereas there are only two outer automorphisms (id and ω) of the affine algebra. As we have mentioned before, inner automorphisms do not modify the charges, and thus the D-branes that preserve the affine symmetry (possibly up to an automorphism) can only account for two summands in the charge group. We want to explain here how to construct symmetry breaking D-branes that carry the remaining charges.

The idea of the construction is recursive. For each of the two different constructions for $\widehat{\mathfrak{su}}(3)$, there are two possible constructions for $\widehat{\mathfrak{su}}(4)$, depending on the choice of the gluing condition for the coset algebra $\widehat{\mathfrak{su}}(4)/\widehat{\mathfrak{su}}(3)$. We call the construction 'straight' if we choose the trivial gluing condition for the coset algebra, and 'twisted' if the gluing condition is the one that is induced from the outer automorphism ω. Thus we get in total four different constructions for SU(4). For each of these four different constructions for SU(4) we have again two different constructions for SU(5), depending on the choice of the gluing condition on the coset $\widehat{\mathfrak{su}}(5)/\widehat{\mathfrak{su}}(4)$, leading to a total of eight constructions for SU(5), etc. Thus, for SU(n), we recursively obtain 2^{n-2} different types of D-branes, which can precisely account for the 2^{n-2} $\mathbb{Z}_d$-summands in the charge group.

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More specifically, this construction leads to boundary conditions that preserve the symmetry algebra

$$\mathcal{W} = \widehat{\mathfrak{su}}(n)/\widehat{\mathfrak{su}}(n-1) \oplus \widehat{\mathfrak{su}}(n-1)/\widehat{\mathfrak{su}}(n-2) \oplus \cdots \oplus \widehat{\mathfrak{su}}(4)/\widehat{\mathfrak{su}}(3) \oplus \widehat{\mathfrak{su}}(3)\,, \tag{2.6}$$

where the coset algebras $\widehat{\mathfrak{su}}(l)/\widehat{\mathfrak{su}}(l-1)$ come from the standard embeddings of $\widehat{\mathfrak{su}}(l-1)$ into $\widehat{\mathfrak{su}}(l)$. More precisely, the embedding is induced by the usual embedding of $\mathfrak{su}(l-1)$ into $\mathfrak{su}(l)$, where the generators of $\mathfrak{su}(l-1)$ describe the upper-left block in $\mathfrak{su}(l)$. They have the nice property that their selection rules are trivial, i.e. every integrable highest weight representation $\mathcal{H}_{\lambda_{l-1}}$ of $\widehat{\mathfrak{su}}(l-1)$ appears in every integrable highest weight representation $\mathcal{H}_{\lambda_l}$ of $\widehat{\mathfrak{su}}(l)$

$$\mathcal{H}_{\lambda_l} \cong \bigoplus_{\lambda_{l-1}\in P^+_{l-1}} \mathcal{H}_{(\lambda_l,\lambda_{l-1})} \otimes \mathcal{H}_{\lambda_{l-1}}\,, \tag{2.7}$$

where the irreducible (non-trivial) representations of the coset algebra are denoted by $\mathcal{H}_{(\lambda_l,\lambda_{l-1})}$. Thus the representations of $\widehat{\mathfrak{su}}(n)$ decompose as

$$\mathcal{H}_{\lambda_n} \cong \bigoplus_{\lambda_i\in P^+_i,\; 3\le i<n} \mathcal{H}_{(\lambda_n,\lambda_{n-1})} \otimes \mathcal{H}_{(\lambda_{n-1},\lambda_{n-2})} \otimes \cdots \otimes \mathcal{H}_{(\lambda_4,\lambda_3)} \otimes \mathcal{H}_{\lambda_3}\,. \tag{2.8}$$

We label such representations by $\hat\lambda = (\lambda_n, \lambda_{n-1}, \ldots, \lambda_3) \in P^+_n \times \cdots \times P^+_3 =: \mathcal{P}$.

As explained above, we want to choose different combinations of gluing conditions for $\mathcal{W}$. Each coset algebra $\widehat{\mathfrak{su}}(l)/\widehat{\mathfrak{su}}(l-1)$ has a non-trivial automorphism, which is induced by the outer automorphism ω (complex conjugation) of $\widehat{\mathfrak{su}}(l)$. By abuse of notation, we will denote it by the same symbol ω. This automorphism leaves the subalgebra $\widehat{\mathfrak{su}}(l-1)$ invariant and actually coincides with the outer automorphism of $\widehat{\mathfrak{su}}(l-1)$ when restricted to it. Thus, we will only use one symbol, ω, for all of the outer automorphisms.

Taking into account that there are also two different gluing conditions for $\widehat{\mathfrak{su}}(3)$, there are two different gluing conditions for each summand of $\mathcal{W}$. We can therefore describe the entire gluing condition on $\mathcal{W}$ by $\hat\omega = (\omega_n, \ldots, \omega_3)$, where each ω_l is either the trivial automorphism id, or the outer automorphism ω. For reasons that will become clear momentarily, it is convenient to choose the convention that $\hat\omega$ describes the gluing condition $\omega_n\,\omega_{n-1}\cdots\omega_l$ on $\widehat{\mathfrak{su}}(l)/\widehat{\mathfrak{su}}(l-1)$ (or on $\widehat{\mathfrak{su}}(3)$ in the case of $l=3$). Here the product of the ω_m is simply the product of the two commuting automorphisms id and ω with the relation $\omega^2 = \mathrm{id}$. So for example, for $\omega_n = \omega$, $\omega_{n-1} = \mathrm{id}$, we have $\omega_n\,\omega_{n-1} = \omega$, etc.

To construct the corresponding boundary states we now have to determine the possible Ishibashi states $|\hat\lambda\rangle\rangle^{\hat\omega}$ for a given gluing condition $\hat\omega$. Because of (2.8) one easily sees that there is exactly one such Ishibashi state for all $\hat\lambda$ for which

$$\begin{aligned} \omega_n(\lambda_n) &= \bar\lambda^*_n = \lambda_n\,, \\ \omega_n\cdots\omega_{l+1}(\lambda_l) &= \bar\lambda^*_l = \omega_n\cdots\omega_l(\lambda_l) \qquad \text{for } n > l \ge 3\,, \end{aligned} \tag{2.9}$$

where $\bar\lambda_l$ denotes the corresponding representation of the right movers, and $*$ denotes charge-conjugation. This simplifies to (here we see the advantage of our convention regarding the gluing condition)

$$\omega_l(\lambda_l) = \lambda_l \qquad \text{for } n \ge l \ge 3\,, \tag{2.10}$$

which are just the conditions for a representation of $\widehat{\mathfrak{su}}(l)$ to be invariant under the twist ω_l. It is convenient to introduce signs ϵ_l that are defined to be $+1$ if $\omega_l = \mathrm{id}$ and -1 if $\omega_l = \omega$; the solution of the above conditions are then exactly given by the $\hat{\lambda} \in P_n^{\epsilon_n} \times \cdots \times P_3^{\epsilon_3} =: \mathcal{P}^{\hat{\epsilon}}$. Thus, for each $\hat{\lambda} \in \mathcal{P}^{\hat{\epsilon}}$ we obtain an Ishibashi state $|\hat{\lambda}\rangle\rangle^{\hat{\omega}}$ of gluing type $\hat{\omega}$, which can be used to construct the boundary states.

The construction of the boundary states can now be done as in [13, 14]. This is to say, we construct boundary states for each $n-2$ tuple $\hat{\rho} = (\rho_n, \ldots, \rho_3)$, where ρ_l labels an untwisted representation of $\widehat{\mathfrak{su}}(l)$ if $\epsilon_l = +1$, and a twisted representation otherwise. They are explicitly given as

$$\|\hat{\rho}\rangle\rangle^{\hat{\omega}} = \sum_{\hat{\lambda} \in \mathcal{P}^{\hat{\epsilon}}} \sqrt{S_{0\lambda_n}} \prod_{l=3}^{n} B_{\rho_l \lambda_l} |\hat{\lambda}\rangle\rangle^{\hat{\omega}} , \tag{2.11}$$

where the coefficients are

$$B_{\rho_l \lambda_l} = \begin{cases} \dfrac{S_{\rho_l \lambda_l}}{S_{0\lambda_l}} & \text{if } \epsilon_l = 1 \\ \dfrac{\psi_{\rho_l \lambda_l}}{S_{0\lambda_l}} & \text{if } \epsilon_l = -1 , \end{cases} \tag{2.12}$$

and where S and ψ denote the untwisted and twisted S-matrices of $\widehat{\mathfrak{su}}(l)$, respectively (there should never be any confusion as to which algebra they belong). Thinking of this construction recursively, this boundary state is obtained by using the straight construction in going from $\widehat{\mathfrak{su}}(l-1)$ to $\widehat{\mathfrak{su}}(l)$ if $\epsilon_l = +1$, and the twisted construction otherwise.

The open string spectrum corresponding to these D-branes consists of representations of $\mathcal{W}$, which are labelled by the tuples

$$(\nu, \sigma) = (\nu_n, \sigma_{n-1}, \nu_{n-1}, \sigma_{n-2}, \ldots, \nu_3, \sigma_3) \in P_n^+ \times P_{n-1}^+ \times P_{n-1}^+ \times \cdots \times P_3^+ \times P_3^+ . \tag{2.13}$$

In particular, following the same arguments as in [13, 14], we obtain the open string spectrum between the D-branes corresponding to $\|\hat{\rho}\rangle\rangle^{\hat{\omega}}$ and $\|\hat{\rho}'\rangle\rangle^{\hat{\omega}}$

$$\mathcal{N}_{\hat{\rho}(\nu,\sigma)}{}^{\hat{\rho}'} = \mathcal{N}^{\epsilon_n}_{\rho_n \nu_n}{}^{\rho'_n} \prod_{l \neq n} \left(\sum_{c_l \in P_l^+} N_{\nu_l \sigma_l^*}{}^{c_l} \, \mathcal{N}^{\epsilon_l}_{\rho_l c_l}{}^{\rho'_l} \right) , \tag{2.14}$$

where we write $\mathcal{N}^-$ for the twisted fusion coefficients (the NIM-rep) and $\mathcal{N}^+$ for the untwisted fusion coefficients.

In fact, it is not difficult to show that these numbers define a NIM-rep of the symmetry algebra $\mathcal{W}$. Since the NIM-rep is a product of factors that involve ρ_l, ρ'_l, it is sufficient to show this separately for each of these factors. (In the following we drop the index l.) Then we compute

$$\begin{aligned} \sum_{z \in P^\epsilon} \sum_{c \in P^+} N_{\nu\sigma^*}{}^c \mathcal{N}^\epsilon_{xc}{}^z \sum_{d \in P^+} N_{\nu'\sigma'^*}{}^d \mathcal{N}^\epsilon_{zd}{}^y &= \sum_{e \in P^+} \mathcal{N}^\epsilon_{xe}{}^y \sum_{c,d \in P^+} N_{\nu\sigma^*}{}^c \, N_{\nu'\sigma'^*}{}^d \, N_{cd}{}^e \\ &= \sum_{e \in P^+} \mathcal{N}^\epsilon_{xe}{}^y \sum_{c,d' \in P^+} N_{\nu\sigma^*}{}^c \, N_{\nu'd'}{}^e \, N_{\sigma'^*c}{}^{d'} \\ &= \sum_{e \in P^+} \mathcal{N}^\epsilon_{xe}{}^y \sum_{c',d' \in P^+} N_{c'\nu}{}^{d'} \, N_{\sigma^*\sigma'^*}{}^{c'} \, N_{d'\nu'}{}^e \end{aligned}$$

$$= \sum_{e \in P^+} \mathcal{N}^{\epsilon}_{xe}{}^{y} \sum_{c', d'' \in P^+} N_{c'^* d''}{}^{e}\, N_{\sigma\sigma'}{}^{c'}\, N_{\nu\nu'}{}^{d''} ,$$

where we have used the NIM-rep property of each $\mathcal{N}^{\pm}$,

$$\sum_{x' \in P^{\pm}} \mathcal{N}^{\pm}_{x\,b}{}^{x'}\, \mathcal{N}^{\pm}_{x'c}{}^{y} = \sum_{f \in P^+} \mathcal{N}^{\pm}_{xf}{}^{y}\, N_{bc}{}^{f} , \qquad (2.15)$$

as well as $N_{\sigma^*\sigma'^*}{}^{c'} = N_{\sigma\sigma'}{}^{c'^*}$. This establishes that we indeed get a NIM-rep for the symmetry algebra $\mathcal{W}$, namely

$$\sum_{\hat{z} \in \mathcal{P}^{\hat{\epsilon}}} \mathcal{N}_{\hat{x}(\nu,\sigma)}{}^{\hat{z}}\, \mathcal{N}_{\hat{z}(\nu',\sigma')}{}^{\hat{y}} = \sum_{(d,c)} N_{(\nu,\sigma)(\nu',\sigma')}{}^{(d,c)}\, \mathcal{N}_{\hat{x}(d,c)}{}^{\hat{y}} , \qquad (2.16)$$

where the fusion rules of the algebra $\mathcal{W}$ are given as

$$N_{(\nu,\sigma)(\nu',\sigma')}{}^{(d,c)} = N_{\nu_n\nu'_n}{}^{d_n} \prod_{l \neq n} \left(N_{\nu_l\nu'_l}{}^{d_l}\, N_{\sigma_l\sigma'_l}{}^{c_l} \right) . \qquad (2.17)$$

In order to calculate the charge group carried by these branes, we need to know the NIM-rep when restricted to the untwisted highest weight representations of $\mathfrak{su}(n)$, since we then get the identity

$$\dim(\lambda)\, q_{\hat{\rho}} = \sum_{\hat{\rho}'} \mathcal{N}_{\hat{\rho}(\lambda,\mathbf{0})}{}^{\hat{\rho}'}\, q_{\hat{\rho}'} , \qquad (2.18)$$

where $(\lambda, \mathbf{0})$ is the representation of $\mathcal{W}$ for which $\nu_n = \lambda$, with all other representations equal to zero. (This is the only natural way in which $\mathfrak{su}(n)$ 'sits inside' $\mathcal{W}$.) What is remarkable, and what allows one to determine the charge group in this case, is that the NIM-rep simplifies for these special weights to

$$\mathcal{N}_{\hat{\rho}(\lambda,\mathbf{0})}{}^{\hat{\rho}'} = \mathcal{N}^{\epsilon_n}_{\rho_n\lambda}{}^{\rho'_n}\, \delta^{(\rho'_{n-1},\ldots,\rho'_3)}_{(\rho_{n-1},\ldots,\rho_3)} . \qquad (2.19)$$

It follows then by the usual argument [8, 9, 10, 11] that the branes for a fixed $(\rho_{n-1}, \ldots, \rho_3)$ give rise to a charge group of the correct size $\mathbb{Z}_d$. In addition, there are presumably other flows that are not of this simple type; they will give rise to relations among the branes differing in $\rho_{n-1}, \ldots, \rho_3$. We conjecture that these flows will be sufficient to deduce that this multiplicity is actually trivial, i.e. that all the branes with a given gluing condition $\hat{\omega}$ which differ in all but the first labels actually carry the same charge. This is plausible from the geometric point of view described in the next section, where it is argued that the geometry of the world-volume depends only on the gluing condition $\hat{\omega}$. In particular, as is explained there, the different boundary states where all signs $\epsilon_i = +1$, wrap the homologically trivial class, and therefore the same class as the usual untwisted (symmetric) D-branes of $\mathrm{SU}(n)$. It is believed that their charges are described by one summand of (1.1), and thus the D-branes that differ in $\rho_{n-1}, \ldots, \rho_3$ cannot generate new charges in this case. It then seems plausible that the same will be true for all other collections of ϵ_i as well.

In summary, these arguments suggest that we get one summand of $\mathbb{Z}_d$ for each combination of gluing conditions, thus producing precisely the full charge group.

3. Some comments about geometry

Following the arguments in [14], the world volumes of the D-branes in SU(n) corresponding to the boundary states (2.11) can be described by products of untwisted and twisted conjugacy classes [19, 20, 21]. To be more precise let us denote by

$$\mathcal{C}_l^+(h) := \{ghg^{-1} \,|\, g \in \mathrm{SU}(l)\}\,, \qquad \mathcal{C}_l^-(h) := \{gh\bar{g}^{-1} \,|\, g \in \mathrm{SU}(l)\} \tag{3.1}$$

the orbits of $h \in \mathrm{SU}(n)$ under the untwisted and twisted adjoint actions of SU(l), which we take to be embedded into SU(n) in the same way as above. Here $\bar{g}$ denotes the complex conjugate of g. Then the world volume corresponding to $\|\hat{\rho}\rangle\rangle^{\hat{\omega}}$ is given by

$$\mathcal{C}_n^{\epsilon_n}(h_{\rho_n}) \cdots \mathcal{C}_n^{\epsilon_3}(h_{\rho_3})\,, \tag{3.2}$$

where $h_{\rho_l} = \exp\left(\frac{2\pi i}{k+l}(\rho_l + \xi)\right) \in \mathrm{SU}(l) \subset \mathrm{SU}(n)$ with ξ the Weyl vector of SU(l) and k the level of the underlying WZW model.

In principle we now have to determine the homology classes of SU(n) represented by such world volumes. In general, this seems to be quite difficult, and we will not attempt to do so here. Instead, we will restrict to some simple special cases, and show that they give rise to linearly independent homology classes.

In the geometric limit $k \to \infty$, all the points h_{ρ_l} converge to the identity e. In particular the untwisted conjugacy classes $\mathcal{C}_l^+(g_{\rho_l})$ therefore degenerate to points, whereas the twisted conjugacy classes $\mathcal{C}_l^-(h_{\rho_l})$ converge to the images of the symmetric spaces SU(l)/SO(l) in SU(n) under the Cartan embedding $[g] \mapsto g\bar{g}^{-1} = gg^t$.

Thus, the world volume corresponding to the boundary states with trivial gluing condition $\hat{\omega}$, i.e. $\epsilon_i = 1$ for all i, is a point. To describe the homology classes represented by D-branes with other gluing conditions, let us recall some facts about the topology of SU(n). As was mentioned before, the homology of SU(n) is isomorphic to the homology of a product of odd dimensional spheres

$$H_*(\mathrm{SU}(n);\mathbb{Z}) \cong H_*\left(S^{2n-1} \times \cdots \times S^3;\mathbb{Z}\right), \tag{3.3}$$

where each of the spheres comes from a homogeneous space $S^{2l-1} \cong \mathrm{SU}(l)/\mathrm{SU}(l-1)$. In particular, the homology of SU(n) is given by the exterior algebra generated by the sphere classes w_{2l-1} of degree $2l-1$, $H_*(\mathrm{SU}(n);\mathbb{Z}) \cong \Lambda[w_3,\ldots,w_{2n-1}]$. (Recall that because of topological obstructions [9] classes containing w_3 cannot be "wrapped" by D-branes but only those in $\Lambda[w_5,\ldots,w_{2n-1}]$.)

For odd l (the analysis for even l seems to be more complicated), the homology of SU(l)/SO(l) is given by $H_*(\mathrm{SU}(l)/\mathrm{SO}(l);\mathbb{Z}) \cong \Lambda[e_5, e_9, \ldots, e_{2l-1}]$ (see e.g. III.6 of [22]). Furthermore, as can be deduced from the considerations in [23], the fundamental class $e_5 \wedge e_9 \wedge \cdots \wedge e_{2l-1}$ of SU(l)/SO(l) is mapped under the Cartan embedding to a class proportional to $w_5 \wedge w_9 \wedge \cdots \wedge w_{2l-1}$, which is thus the class represented by $\mathcal{C}_l^-(e) \subset \mathrm{SU}(n)$. (A special case of this are the twisted D-branes of SU(3) whose geometry has been discussed for example in [9].) In particular, it therefore follows that at least those constructions for which $\epsilon_i = 1$ for all except for one odd i lead to D-branes that wrap homologically inequivalent cycles. Since our construction is in essence recursive, one would then expect that all classes in $\Lambda[w_5, \ldots, w_{2n-1}]$ can be represented by the world volumes of the various D-branes obtained by it. Given the relation of K-theory to homology [9], this would then suggest that these boundary states indeed account for all the K-theory charges.

4. Conclusions

In this paper we have given a construction of a collection of D-branes for $\mathrm{SU}(n)$ that seems to account for all the K-theory charges of $\mathrm{SU}(n)$. In particular, we have argued that the 2^{n-2} different constructions lead to D-branes that wrap homologically different submanifolds of $\mathrm{SU}(n)$. Given the close relation between homology and K-theory, it is then very plausible that the corresponding D-branes carry inequivalent K-theory charges. We have given evidence that each of the 2^{n-2} different constructions leads to one summand of $\mathbb{Z}_d$ in the K-theory charge group. Our construction therefore seems to account for all K-theory charges.

The construction is technically very similar indeed to the construction of the D-branes that was proposed in [12]. In particular, the different signs s_m in [12] correspond to choosing the straight or twisted construction for $\widehat{\mathfrak{su}}(m)/\widehat{\mathfrak{su}}(m-1)$, and so $s_m = \epsilon_n \cdots \epsilon_m$. The first sign s_n in that construction determines whether the resulting NIM-rep of $\widehat{\mathfrak{su}}(n)$ is the untwisted fusion rule or the twisted NIM-rep, as in (2.19); its last sign s_1 is immaterial because of the Weyl symmetry, which is also the reason why there is only one construction for SU(2), and a reason we can stop (1.6) at $\widehat{\mathfrak{su}}(3)$. It is therefore very plausible that the branes of [12] lie on the same sheets of moduli space as the D-branes that have been constructed here. In the formulation of [12] the conformal field theory analysis of the charges was more transparent (since the whole NIM-rep could be interpreted in terms of twisted representations of $\widehat{\mathfrak{su}}(n)$), whereas in the description given here the geometrical interpretation is more suggestive. Taken together, the two constructions provide therefore good evidence that either of them gives a construction of the D-branes that carry all the K-theory charges of $\mathrm{SU}(n)$.

In this paper, we have only considered the case of $\mathrm{SU}(n)$, but it should be straightforward (just as for the construction in [12]) to generalise it to other Lie groups. Given the structure of our constructions, it seems likely that the corresponding D-branes will again be numerous enough to account for all K-theory charges. It would be interesting to check this in detail though.

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